An application of proof mining to nonlinear iterations

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Abstract

In this paper we apply methods of proof mining to obtain a highly uniform effective rate of asymptotic regularity for the Ishikawa iteration associated to nonexpansive self-mappings of convex subsets of a class of uniformly convex geodesic spaces. Moreover, we show that these results are guaranteed by a combination of logical metatheorems for classical and semi-intuitionistic systems.

1 Introduction

Proof mining is a paradigm of research concerned with the extraction of hidden finitary and combinatorial content from proofs that make use of highly infinitary principles. This new information is obtained after a logical analysis of the original mathematical proof, using proof-theoretic techniques called *proof interpretations*. In this way one obtains highly uniform effective bounds for results that are more general than the initial ones. While the methods used to obtain these new results come from mathematical logic, their proofs can be written in ordinary mathematics. We refer to Kohlenbach's book [19] for a comprehensive reference for proof mining.

This line of research, developed by Ulrich Kohlenbach in the 90's, has its origins in Kreisel's program of *unwinding of proofs*. Kreisel's idea was to apply proof-theoretic techniques to analyze concrete mathematical proofs and unwind the information hidden in them; see for example [23] and, more recently, [28].

Proof mining has numerous applications to approximation theory, metric fixed point theory, asymptotic behaviour of nonlinear iterations, as well as (nonlinear) ergodic theory, topological dynamics and Ramsey theory. In these applications, Kohlenbach's *monotone* functional interpretation [14] is crucially used, since it systematically transforms any statement in a given proof into a new version for which explicit bounds are provided.

Terence Tao [38] arrived at a proposal of so-called *hard analysis* (as opposed to *soft analysis*), inspired by the finitary arguments used by him and Green [10]

in their proof that there are arithmetic progressions of arbitrary length in the prime numbers, as well as by him alone in a series of papers [37, 39, 40, 41]. As Kohlenbach points out in [17], Tao's hard analysis could be viewed as carrying out, using monotone functional interpretation, analysis on the level of uniform bounds.

For mathematical proofs based on classical logic, general logical metatheorems were obtained by Kohlenbach [16] for important classes of metrically bounded spaces in functional analysis and generalized to the unbounded case by Gerhardy and Kohlenbach [8]. They considered metric, hyperbolic and CAT(0)-spaces, (uniformly convex) normed spaces and inner product spaces also with abstract convex subsets. The metatheorems were adapted to Gromov δ-hyperbolic spaces, $\mathbb{R}$-trees and uniformly convex hyperbolic spaces in [24], complete metric and normed spaces in [19] and uniformly smooth spaces in [20]. The proofs of the metatheorems are based on extensions to the new formal systems of Gödel's Dialectica interpretation combined with negative translation and parametrized versions of majorization.

These logical metatheorems guarantee that one can extract bounds from classical proofs of $\forall\exists$-sentences and that these bounds are independent from parameters satisfying weak local boundedness conditions. Thus, the metatheorems can be used to conclude the existence of uniform bounds without having to carry out the proof analysis: we have to verify only that the statement has the right logical form and that the proof can be formalized in our system.

Gerhardy and Kohlenbach [7] obtained similar logical metatheorems for proofs in semi-intuitionistic systems, that is proofs based on intuitionistic logic enriched with noneffective principles, such as comprehension in all types for arbitrary negated or $\exists$-free formulas. The proofs of these metatheorems use monotone modified realizability, a monotone version of Kreisel's modified realizability [22]. A great benefit of this setting is that there are basically no restrictions on the logical complexity of mathematical theorems for which bounds can be extracted.

The goal of this paper is to present an application of proof mining to the asymptotic behaviour of Ishikawa iterations for nonexpansive mappings.

Let X be a normed space, $C \subseteq X$ a convex subset and $T : C \to C$. The *Ishikawa iteration* starting with $x \in C$ was introduced in [13] as follows:

$$x_0 := x \in C, \quad x_{n+1} = (1-\lambda_n)x_n + \lambda_n T((1-s_n)x_n + s_n T x_n),$$

where $(\lambda_n), (s_n)$ are sequences in $[0,1]$. The well-known Krasnoselski-Mann iteration [21, 29] is obtained as a special case, by taking $s_n = 0$ for all $n \in \mathbb{N}$.

Ishikawa proved that for convex compact subsets C of Hilbert spaces and Lipschitzian pseudo-contractive mappings T, this iteration converges strongly towards a fixed point of T, provided that the sequences (λ_n) and (s_n) satisfy some assumptions.

In the following we consider the Ishikawa iteration for nonexpansive map-

pings and sequences $(\lambda_n), (s_n)$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) \text{ diverges, } \limsup_{n\to\infty} s_n < 1 \text{ and } \sum_{n=0}^{\infty} s_n(1-\lambda_n) \text{ converges.} \qquad (1)$$

Tan and Xu [36] proved the weak convergence of the Ishikawa iteration in uniformly convex Banach spaces X which satisfy Opial's condition or whose norm is Fréchet differentiable, generalizing in this way a seminal result of Reich [32] for the Krasnoselski-Mann iteration. Dhompongsa and Panyanak [6] obtained the Δ-convergence of the Ishikawa iteration in CAT(0) spaces. Δ-convergence is a concept of weak convergence in metric spaces introduced by Lim [27].

One of the most important properties of any iteration associated to a nonlinear mapping is asymptotic regularity, defined by Browder and Petryshyn [3] for the Picard iteration. The *Ishikawa iteration* (x_n) is said to be *asymptotically regular* if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. A rate of convergence of $(\|x_n - Tx_n\|)$ towards 0 will be called a *rate of asymptotic regularity* of (x_n). Asymptotic regularity is the first property one gets before proving the weak or strong convergence of the iteration towards a fixed point of the mapping. Thus, the following asymptotic regularity result is implicit in the proof of Tan and Xu.

Theorem 1.1. *Let X be a uniformly convex Banach space, $C \subseteq X$ a bounded closed convex subset and $T : C \to C$ be nonexpansive. Assume that $(\lambda_n), (s_n)$ satisfy (1).*

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x \in C$.

The main result of this paper (Theorem 2.1) is a finitary version of a generalization of the above result to convex subsets of a class of uniformly convex geodesic spaces, the so-called UCW-hyperbolic spaces. Furthermore, we do not require that the space X to be complete and that C to be closed and bounded. We replace the hypothesis that C is bounded with the much weaker one that T has approximate fixed points in a b-neighborhood of x for some $b > 0$, by which we mean that for all $\varepsilon > 0$ there exists $y \in C$ such that $\|x - y\| \le b$ and $\|x - Tx\| < \varepsilon$. We obtain a highly uniform explicit rate of asymptotic regularity for this iteration, generalizing the results from [26].

Furthermore, as we shall explain in Sections 3 and 4, the existence of such a rate is guaranteed by a combination of logical metatheorems for the classical and semi-intuitionistic setting. As we use both monotone Dialectica and monotone modified realizability interpretations to give a logical explanation of our results, we think that an interesting direction of research could be to see if the hybrid functional interpretation [11, 30] can be used instead.

Notation: $\mathbb{N} = \{0, 1, 2 \ldots, \}$ and $[m, n] = \{m, m+1, \ldots, n-1, n\}$ for any $m, n \in \mathbb{N}, m \le n$.

2 Main result

A W-*hyperbolic space* is a structure (X, d, W), where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a *convexity* mapping satisfying the following axioms:

$$\begin{array}{ll}
(W1) & d(z, W(x, y, \lambda)) \le (1-\lambda)d(z, x) + \lambda d(z, y), \\
(W2) & d(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot d(x, y), \\
(W3) & W(x, y, \lambda) = W(y, x, 1-\lambda), \\
(W4) & d(W(x, z, \lambda), W(y, w, \lambda)) \le (1-\lambda)d(x, y) + \lambda d(z, w).
\end{array}$$

Takahashi [35] initiated in the 70's the study of *convex metric spaces* as structures (X, d, W) satisfying (W1). The notion of W-hyperbolic space defined above was introduced by Kohlenbach [16]. We refer to [19, p.384] for a very nice discussion on these spaces and related structures. First examples of W-hyperbolic spaces are normed spaces; just take $W(x, y, \lambda) = (1-\lambda)x + \lambda y$. A very important class of W-hyperbolic spaces are Busemann's non-positively curved spaces [4, 5], extensively studied in the monograph [31].

Given $x, y \in X$ and $\lambda \in [0, 1]$, we shall use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x, y, \lambda)$. A nonempty subset $C \subseteq X$ is said to be *convex* if $(1-\lambda)x \oplus \lambda y \in C$ for all $x, y \in C$ and all $\lambda \in [0, 1]$.

Uniform convexity can be defined in the setting of W-hyperbolic spaces following Goebel and Reich's definition for the Hilbert ball [9, p.105]. A W-hyperbolic space (X, d, W) is said to be *uniformly convex* [25] if there exists a mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ such that for all $r > 0, \varepsilon \in (0, 2]$ and all $a, x, y \in X$,

$$\left.\begin{array}{l} d(x, a) \le r \\ d(y, a) \le r \\ d(x, y) \ge \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \le (1 - \eta(r, \varepsilon))r.$$

The mapping η is said to be a *modulus of uniform convexity*. We use the notation (X, d, W, η) for a uniformly convex W-hyperbolic space with modulus η.

Uniformly convex W-hyperbolic spaces (X, d, W, η) with η being decreasing with r (for a fixed ε) are called UCW-*hyperbolic spaces*, following [26]. Obviously, uniformly convex Banach spaces are UCW-hyperbolic spaces with a modulus η that does not depend on r at all. Other examples of UCW-hyperbolic space are CAT(0) spaces, important structures in geometric group theory (see, e.g. [2]). As the author remarked in [25], CAT(0) spaces have a modulus of uniform convexity $\eta(\varepsilon) = \dfrac{\varepsilon^2}{8}$, quadratic in ε.

If (X, d) is a metric space and $C \subseteq X$ is a nonempty subset, then a mapping $T : C \to C$ is said to be *nonexpansive* if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in C$. We denote with $Fix(T)$ the set of fixed points of T. For $x \in X$ and $b, \delta > 0$, let

$$Fix_\delta(T, x, b) := \{y \in C \mid d(y, x) \le b \text{ and } d(y, Ty) < \delta\}.$$

If $Fix_\delta(T,x,b) \neq \emptyset$ for all $\delta > 0$, we say that T has *approximate fixed points* in a b-neighborhood of x.

We recall now some notions needed for expressing our quantitative results. Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If (a_n) converges towards a, then a mapping $\gamma : (0,\infty) \to \mathbb{N}$ is said to be

- a *Cauchy modulus* of (a_n) if $|a_{\gamma(\varepsilon)+n} - a_{\gamma(\varepsilon)}| < \varepsilon$ for all $\varepsilon > 0, n \in \mathbb{N}$;

- a *rate of convergence* of (a_n) if $|a_{\gamma(\varepsilon)+n} - a| < \varepsilon$ for all $\varepsilon > 0, n \in \mathbb{N}$.

If the series $\sum_{n=0}^{\infty} a_n$ converges, then a Cauchy modulus of the sequence (s_n) of partial sums, $s_n := \sum_{i=0}^{n} a_i$, is said to be a *Cauchy modulus* of $\sum_{n=0}^{\infty} a_n$. If $\sum_{n=0}^{\infty} a_n$ diverges, then a *rate of divergence* of $\sum_{n=0}^{\infty} a_n$ is a mapping $\theta : \mathbb{N} \to \mathbb{N}$ satisfying. $\sum_{i=0}^{\theta(n)} a_i \geq n$ for all $n \in \mathbb{N}$.

The main result of the paper is the following quantitative version of the generalization to UCW-hyperbolic spaces of Theorem 1.1, giving an effective and highly uniform rate of asymptotic regularity for the Ishikawa iterations.

Theorem 2.1. *Let (X,d,W,η) be a UCW-hyperbolic space, $C \subseteq X$ a convex subset and $T : C \to C$ be a nonexpansive mapping.*

Assume that $(\lambda_n), (s_n)$ are sequences in $[0,1]$ satisfying the following properties

(i) $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n)$ *with rate of divergence* $\theta : \mathbb{N} \to \mathbb{N}$*;*

(ii) $\limsup_n s_n < 1$ *with* $L, N_0 \in \mathbb{N}$ *satisfying* $s_n \leq 1 - \dfrac{1}{L}$ *for all* $n \geq N_0$*;*

(iii) $\sum_{n=0}^{\infty} s_n(1-\lambda_n)$ *converges with Cauchy modulus* γ *and* $M > 0$ *is an upper bound on* $\sum_{n=0}^{\infty} s_n(1-\lambda_n)$.

Suppose that $x \in C, b > 0$ are such that

$$d(x,Tx) \leq b \ \textit{and} \ \forall \delta > 0 (Fix_\delta(T,x,b) \neq \emptyset). \tag{2}$$

Let (x_n) be the Ishikawa iteration starting with x, defined by

$$x_0 := x \in C, \quad x_{n+1} = (1-\lambda_n)x_n \oplus \lambda_n T((1-s_n)x_n \oplus s_n T x_n). \tag{3}$$

Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and moreover*

$$\forall \varepsilon > 0 \forall n \geq \Phi(\varepsilon, \eta, b, \theta, L, N_0, \gamma, M)\Big(d(x_n, Tx_n) < \varepsilon\Big), \tag{4}$$

where

$$\Phi := \Phi(\varepsilon, \eta, b, N_0, L, \theta, \gamma, M) = \theta(P + \gamma_0 + 1 + N_0), \tag{5}$$

with

$$\gamma_0 = \gamma\left(\frac{\varepsilon}{4b\exp(2M)}\right) \quad \text{and} \quad P = \left\lceil \frac{2L(b+1)}{\varepsilon \cdot \eta\left(b+1, \dfrac{\varepsilon}{2L(b+1)}\right)} \right\rceil.$$

We emphasize that the bound Φ does not depend on X, C, T, x except for b and the modulus η of uniform convexity.

Since bounded closed convex subsets C of Banach spaces satisfy (2) for all $x \in C$ by taking b as the diameter of C, it is obvious that our main result implies Theorem 1.1. Furthermore, Theorem 2.1 generalizes with slightly changed bounds the main result of [25], where the stronger hypothesis that T has fixed points was assumed.

We give the proof of our result in Section 5. Before doing this, we explain in the next two sections that the extractability of such a uniform rate of asymptotic regularity is guaranteed by logical metatheorems.

3 Logical metatheorems for UCW-hyperbolic spaces

In the following we give adaptations to UCW-hyperbolic spaces of general logical metatheorems for W-hyperbolic spaces proved by Gerhardy and Kohlenbach for classical systems in [8] and for intuitionistic systems in [7].

Let $\mathcal{A}^\omega$ be the system of *weakly extensional* classical analysis, which goes back to Spector [34]. It is formulated in the language of functionals of finite types and consists of $\mathbf{WE}-\mathbf{PA}^\omega$, the weakly extensional Peano arithmetic in all finite types, the axiom schema $\mathbf{QF}-\mathbf{AC}$ of quantifier-free axiom of choice and the axiom schema $\mathbf{DC}^\omega$ of dependent choice in all finite types. Full second order arithmetic in the sense of reverse mathematics [33] is contained in $\mathcal{A}^\omega$ if we identify subsets of $\mathbb{N}$ with their characteristic functions. We refer the reader to [19] for all the undefined notions related to the system $\mathcal{A}^\omega$, including the representation of real numbers in this system. As a consequence of this representation, the relations $=_\mathbb{R}$, $\leq_\mathbb{R}$ are given by Π^1_0 predicates, while $<_\mathbb{R}$ is given by a Σ^1_0 predicate.

The theory $\mathcal{A}^\omega[X, d]_{-b}$ for metric spaces is defined in [8] by extending $\mathcal{A}^\omega$ to the set $\mathbf{T}^X$ of all finite types over the ground types 0 and X and by adding two new constants 0_X of type X and d_X of type $X \to X \to 1$ together with

axioms expressing the fact that d_X represents a pseudo-metric. One defines the equality $=_X$ between objects of type X as follows:

$$x =_X y := d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}}.$$

Then d_X represents a metric on the set of equivalence classes generated by $=_X$.

We use the subscript $_{-b}$ here and for the theories defined in the sequel in order to be consistent with the notations from [19].

The theory $\mathcal{A}^\omega[X, d, W]_{-b}$ for W-hyperbolic spaces results from $\mathcal{A}^\omega[X, d]_{-b}$ by adding a new constant W_X of type $X \to X \to 1 \to X$ together with the formalizations of the axioms (W1)-(W4).

The theory $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ of UCW-hyperbolic spaces was defined in [24] as an extension of $\mathcal{A}^\omega[X, d, W]_{-b}$ obtained by adding a constant η_X of type $0 \to 0 \to 0$, together with three universal axioms expressing that η_X represents a modulus of uniform convexity decreasing with r for fixed ε and that η_X is a function having the first argument a rational number on the level of codes.

$$\begin{aligned}
(A1)\quad & \forall r^0, k^0 \forall x^X, y^X, a^X \big(d_X(x, a) <_{\mathbb{R}} r \wedge d_X(y, a) <_{\mathbb{R}} r \wedge \\
& d_X(W_X(x, y, 1/2), a) >_{\mathbb{R}} \left(1 - 2^{-\eta_X(r,k)}\right) \cdot_{\mathbb{R}} r \to d_X(x, y) \leq_{\mathbb{R}} 2^{-k} \cdot_{\mathbb{R}} r\big),
\end{aligned}$$

$$(A2)\quad \forall r_1^0, r_2^0, k^0 (r_1 \leq_{\mathbb{Q}} r_2 \to \eta_X(r_1, k) \geq_0 \eta_X(r_2, k)),$$

$$(A3)\quad \forall r^0, k^0 (\eta_X(r, k) =_0 \eta_X(c(r), k)),$$

where $c(n) := \min p \leq_0 n[p =_{\mathbb{Q}} n]$ is the canonical representative for rational numbers (see [15] for details).

If X is a nonempty set, the full-theoretic type structure $S^{\omega,X} := \langle S_\rho \rangle_{\rho \in \mathbf{T}^X}$ over 0 and X is defined by

$$S_0 := \mathbb{N}, \quad S_X := X, \quad S_{\rho \to \tau} := S_\tau^{S_\rho},$$

where $S_\tau^{S_\rho}$ is the set of all set-theoretic functions $S_\rho \to S_\tau$.

Let (X, d, W, η) be a UCW-hyperbolic space. $S^{\omega,X}$ becomes a model of $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ by letting the variables of type ρ range over S_ρ, giving the natural interpretations to the constants of $\mathcal{A}^\omega$, interpreting 0_X by an arbitrary element of X, the constants d_X and W_X as specified in [16] and η_X by $\eta_X(r, k) := \eta(c(r), k)$.

We say that a sentence in the language $\mathcal{L}(\mathcal{A}^\omega[X, d, UCW, \eta]_{-b})$ holds in a nonempty UCW-hyperbolic spaces (X, d, W, η) if it is true in all models of $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ obtained from $S^{\omega,X}$ as above.

From now on, in order to improve readability, we shall usually omit the subscripts $_{\mathbb{N}}, _{\mathbb{R}}, _{\mathbb{Q}}, _X$ excepting the cases where such an omission could create confusions. We shall use $\mathbb{N}$ instead of 0 and $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N} \to \mathbb{N}$ instead of 1 and, moreover, write $n \in \mathbb{N}, f : \mathbb{N} \to \mathbb{N}, x \in X, T : X \to X$ instead of $n^0, f^1, x^X, T^{X \to X}$.

The notion of *majorizability* was originally introduced by Howard [12], and subsequently modified by Bezem [1]. For any type $\rho \in \mathbf{T}^X$, we define the type $\widehat{\rho} \in \mathbf{T}$, obtained by replacing all occurrences of the type X in ρ by $\mathbb{N}$. Based on Bezem's notion of *strong majorizability s-maj* [1], Gerhardy and Kohlenbach [8]

defined, for every parameter a of type X, an *a-majorization* relation $\gtrsim_\rho^a$ between objects of type $\rho \in \mathbf{T}^X$ and their majorants of type $\widehat{\rho} \in \mathbf{T}$ as follows:

(i) $x^* \gtrsim_{\mathbb{N}}^a x :\equiv x^* \geq x$ for $x, x^* \in \mathbb{N}$;

(ii) $x^* \gtrsim_X^a x :\equiv (x^*)_{\mathbb{R}} \geq_{\mathbb{R}} d(x,a)$ for $x^* \in \mathbb{N}, x \in X$;

(iii) $x^* \gtrsim_{\rho\to\tau}^a x :\equiv \forall y^*, y(y^* \gtrsim_\rho^a y \to x^*y^* \gtrsim_\tau^a xy) \wedge \forall z^*, z(z^* \gtrsim_{\hat{\rho}}^a z \to x^*z^* \gtrsim_{\hat{\tau}}^a x^*z)$.

Restricted to the types $\mathbf{T}$, the relation $\gtrsim^a$ coincides with strong majorizability *s-maj* and, hence, for $\rho \in \mathbf{T}$ one writes *s-maj*$_\rho$ instead of $\gtrsim_\rho^a$, as in this case the parameter a is irrelevant.

If $t^* \gtrsim^a t$ for terms t^*, t, we say that t^* *a-majorizes* t or that t is *a-majorized* by t^*. A term t is said to be *majorizable* if it has an a-majorant for some $a \in X$. One can prove that t is majorizable if and only if it has an a-majorant for all $a \in X$ (see, e.g., [19, Lemma 17.78]).

Lemma 3.1. *Let $T : X \to X$. The following are equivalent.*

(i) T is majorizable;

(ii) for all $x \in \mathbb{N}$ there exists $\Omega : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, y \in X\Big(d(x,y) < n \to d(x,Ty) \leq \Omega(n)\Big); \tag{6}$$

(iii) for all $x \in \mathbb{N}$ there exists $\Omega : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, y \in X\Big(d(x,y) \leq n \to d(x,Ty) \leq \Omega(n)\Big). \tag{7}$$

Proof. T is majorizable if and only if T is x-majorizable for each $x \in X$ if and only if for each $x \in X$ there exists a function $T^* : \mathbb{N} \to \mathbb{N}$ such that T^* is monotone and satisfies

$$\forall n \in \mathbb{N}\, \forall y \in X\Big(d(x,y) \leq n \to d(x,Ty) \leq T^*n\Big).$$

$(i) \Rightarrow (iii)$ is obvious: take $\Omega := T^*$. For the implication $(iii) \Rightarrow (i)$, given, for $x \in X$, Ω satisfying (7), define $T^*n := \max\limits_{k \leq n} \Omega(k)$.

$(iii) \Rightarrow (ii)$ is again obvious. For the converse implication, given Ω satisfying (6) define $\tilde{\Omega}(n) := \Omega(n+1)$. Then $\tilde{\Omega}$ satisfies (7) $\square$

In the sequel, given a majorizable function $T : X \to X$ and $x \in X$, an Ω satisfying (7) will be called a *modulus of majorizability at x* of T; we say also that T is *x-majorizable with modulus* Ω. We gave in the lemma above the equivalent condition (6) for logical reasons: since $<_{\mathbb{R}}$ is a Σ_1^0 predicate and $\leq_{\mathbb{R}}$ is a Π_1^0 predicate, the formula in (6) is universal.

The following lemma shows that natural classes of mappings in metric or W-hyperbolic spaces are majorizable; we refer to [19, Corollary 17.55] for the proof.

Lemma 3.2. *Let (X, d) be a metric space.*

(i) If (X, d) is bounded with diameter d_X, then any function $T : X \to X$ is majorizable with modulus of majorizability $\Omega(n) := \lceil d_X \rceil$ for each $x \in X$.

(ii) If $T : X \to X$ is L-Lipschitz, then T is majorizable with modulus at x given by $\Omega(n) := n + L^*b$, where $b, L^* \in \mathbb{N}$ are such that $d(x, Tx) \leq b$ and $L \leq L^*$. In particular, any nonexpansive mapping is majorizable with modulus $\Omega(n) := n + b$.

(iii) If (X, d, W) is a W-hyperbolic space, then any uniformly continuous mapping $T : X \to X$ is majorizable with modulus $\Omega(n) := n \cdot 2^{\alpha_T(0)} + 1 + b$ at x, where $d(x, Tx) \leq b \in \mathbb{N}$ and α_T is a modulus of uniform continuity of T, i.e. $\alpha_T : \mathbb{N} \to \mathbb{N}$ satisfies

$$\forall x, y \in X \, \forall k \in \mathbb{N} \left(d(x, y) \leq 2^{-\alpha_T(k)} \to d(Tx, Ty) \leq 2^{-k} \right).$$

Given a type $\rho \in \mathbf{T}^X$, we say that

(i) ρ has degree $(0, X)$ if $\rho = X$ or $\rho = \mathbb{N} \to \ldots \to \mathbb{N} \to X$;

(ii) ρ is of degree $(1, X)$ if $\rho = X$ or has the form $\rho = \rho_1 \to \ldots \to \rho_n \to X$, where $n \geq 1$ and each ρ_i has degree ≤ 1 or $(0, X)$;

(iii) ρ has degree 1^* if $\widehat{\rho}$ has degree ≤ 1.

Whenever we write $A(\underline{u})$, we mean that A is a formula in our language which has only the variables $\underline{u}$ free. A formula A is called a $\forall$-*formula* (resp. a $\exists$-*formula*) if it has the form

$$A \equiv \forall \underline{x}^{\underline{\sigma}} A_0(\underline{x}, \underline{a}) \quad (\text{resp. } A \equiv \exists \underline{x}^{\underline{\sigma}} A_0(\underline{x}, \underline{a})),$$

where A_0 is a quantifier free formula and the types in $\underline{\sigma}$ are of degree 1^* or $(1, X)$. We assume in the following that the constant 0_X does not occur in the formulas we consider; this is no restriction, since 0_X is just an arbitrary constant which could have been replaced by any new variable of type X.

The following result is an adaptation of a general logical metatheorem proved first by Kohlenbach [16] for bounded W-hyperbolic spaces, and then generalized to the unbounded case by Gerhardy and Kohlenbach [8].

Theorem 3.3. *[24] Let P be $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$, K be an $\mathcal{A}^\omega$-definable compact metric space, ρ be of degree 1^*, $B_\forall(u, y, z, n)$ be a $\forall$-formula and $C_\exists(u, y, z, N)$ be an $\exists$-formula. Assume that $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ proves that*

$$\forall u \in P \forall y \in K \forall z^\rho \left(\forall n \in \mathbb{N} \, B_\forall \to \exists N \in C_\exists \right).$$

Then one can extract a computable functional $\Phi : P \times \mathbb{N}^{(\mathbb{N} \times \ldots \times \mathbb{N})} \times \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \to \mathbb{N}$ such that the following statement is satisfied in all nonempty UCW-hyperbolic spaces (X, d, W, η):

for all $z \in S_\rho, z^* \in \mathbb{N}^{(\mathbb{N}\times\ldots\times\mathbb{N})}$*, if there exists* $a \in X$ *such that* $z^* \gtrsim_\rho^a z$*, then*

$$\forall u \in P \forall y \in K \left(\forall n \leq \Phi(u, z^*, \eta)\, B_\forall \to \exists N \leq \Phi(u, z^*, \eta)\, C_\exists \right).$$

Remark 3.4. *(i) Instead of single premises* $\forall n B_\forall$ *and single variables* u, y, n *we may have finite conjunctions of premises as well as tuples* $\underline{u} \in P, \underline{y} \in K, \underline{n} \in \mathbb{N}$ *of variables.*

(ii) We can have also $\underline{z}^{\underline{\rho}} = z_1^{\rho_1}, \ldots, z_k^{\rho_k}$ *for types* $\rho_1, \ldots, \rho_k$ *of degree* 1^**. Then in the conclusion is assumed that* $z_i^* \gtrsim_{\rho_i}^a z_i$ *for a* common $a \in X$ *for all* $i = 1, \ldots, k$*. The bound* Φ *depends now on all the a-majorants* $z_1^*, \ldots, z_k^*$*.*

The proof of Theorem 3.3 is based on an extension to $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ of Spector's [34] interpretation of classical analysis $\mathcal{A}^\omega$ using *bar recursion*, combined with a-majorization. Furthermore, the proof of the metatheorem actually provides an extraction algorithm for the functional Φ, which can always be defined in the calculus of bar-recursive functionals. However, as we shall see also in this paper, for concrete applications usually small fragments of $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ are needed to formalize the proof. As a consequence, one gets bounds of primitive recursive complexity and very often exponential or even polynomial bounds.

We give now a very useful corollary of Theorem 3.3.

Corollary 3.5. *Let* P *be* $\mathbb{N}$ *or* $\mathbb{N}^\mathbb{N}$*,* K *an* $\mathcal{A}^\omega$*-definable compact metric space,* $B_\forall(\underline{u}, \underline{y}, x, x^*, T, n)$ *be a* $\forall$*-formula and* $C_\exists(\underline{u}, \underline{y}, x, x^*, T, N)$ *a* $\exists$*-formula. Assume that* $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ *proves that*

$$\forall\, \underline{u} \in P\, \forall\, \underline{y} \in K\, \forall\, x, x^* \in X\, \forall\, T : X \to X\, \forall\, \Omega : \mathbb{N} \to \mathbb{N} \left(T \text{ is } x\text{-majorizable with modulus } \Omega \,\wedge\, \forall n \in \mathbb{N}\, B_\forall \,\to\, \exists N \in \mathbb{N}\, C_\exists \right).$$

Then one can extract a computable functional Φ *satisfying the following statement for all* $\underline{u} \in P$*,* $b \in \mathbb{N}$ *and* $\Omega : \mathbb{N} \to \mathbb{N}$*:*

$$\forall\, \underline{y} \in K\, \forall\, x, x^* \in X\, \forall\, T : X \to X \Big(T \text{ is } x\text{-majorizable with modulus } \Omega \,\wedge\, d(x, x^*) \leq b \,\wedge\, \forall n \leq \Phi(\underline{u}, b, \Omega, \eta)\, B_\forall \to\, \exists N \leq \Phi(\underline{u}, b, \Omega, \eta)\, C_\exists \Big).$$

holds in all nonempty UCW*-hyperbolic spaces* (X, d, W, η)*.*

Proof. The premise "T is x-majorizable with modulus Ω" is a $\forall$-formula, by (6). Furthermore, 0 x-majorizes x, b is an x-majorant for x^*, since $d(x, x^*) \leq b$, and $T^* := \lambda n. \max_{k \leq n} \Omega(k)$ x-majorizes T, by the proof of Lemma 3.1. Apply now Theorem 3.3. ∎

Remark 3.6. *As in the case of Theorem 3.3, instead of single $n \in \mathbb{N}$ and a single premise $\forall n B_\forall$ we could have tuples $\underline{n} = n_1, \ldots, n_k$ and a conjunction of premises $\forall n_1 B^1_\forall \wedge \ldots \wedge \forall n_k B^k_\forall$. In this case, in the premise of the conclusion we shall have $\forall n_1 \leq \Phi\, B^1_\forall \wedge \ldots \wedge \forall n_k \leq \Phi\, B^k_\forall$.*

The following more concrete consequence of Theorem 3.3 shows that, under some conditions, the hypothesis of T having fixed points can be replaced by the weaker one that T has approximate fixed points in a b-neighborhood of x. Its proof is similar with the one of [8, Corollary 4.22].

Corollary 3.7. *Let P be $\mathbb{N}$ or $\mathbb{N}^\mathbb{N}$, K be an $\mathcal{A}^\omega$-definable compact metric space, $B_\forall(\underline{u}, \underline{y}, x, T, n)$ be a $\forall$-formula and $C_\exists(\underline{u}, \underline{y}, x, T, N)$ a $\exists$-formula. Assume that $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ proves that*

$$\forall \underline{u} \in P\, \forall \underline{y} \in K\, \forall x \in X\, \forall T : X \to X\, \forall \Omega : \mathbb{N} \to \mathbb{N}$$
$$\Big(T \text{ is } x\text{-majorizable with mod. } \Omega \wedge Fix(T) \neq \emptyset \wedge \forall n \in \mathbb{N}\, B_\forall \to \exists N \in \mathbb{N}\, C_\exists\Big).$$

It follows that one can extract a computable functional Φ such that for all $\underline{u} \in P, b \in \mathbb{N}$ and $\Omega : \mathbb{N} \to \mathbb{N}$,

$$\forall \underline{y} \in K\, \forall x \in X\, \forall T : X \to X$$
$$\Big(T \text{ is } x\text{-majorizable with modulus } \Omega \wedge \forall \delta > 0\big(Fix_\delta(T, x, b) \neq \emptyset\big) \wedge$$
$$\forall n \leq \Phi(\underline{u}, b, \Omega, \eta) B_\forall \to \exists N \leq \Phi(\underline{u}, b, \Omega, \eta)\, C_\exists\Big).$$

holds in any nonempty UCW-hyperbolic space (X, d, W, η).

Proof. The statement proved in $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ can be written as

$$\forall \underline{u} \in P\, \forall \forall \underline{y} \in K\, \forall x, p \in X\, \forall T : X \to X\, \Omega : \mathbb{N} \to \mathbb{N}$$
$$\Big(T\, x\text{-maj. mod.}\Omega \wedge \forall k \in \mathbb{N}\left(d(p, Tp) \leq_\mathbb{R} 2^{-k}\right) \wedge \forall n \in \mathbb{N} B_\forall \to \exists N \in \mathbb{N}\, C_\exists\Big).$$

We have used the fact that $Fix(T) \neq \emptyset$ is equivalent with $\exists p \in X(Tp =_X p)$ that is further equivalent with $\exists p \in X\, \forall\, k \in \mathbb{N}\big(d(p, Tp) \leq_\mathbb{R} 2^{-k}\big)$, by using the definition of $=_X$ and $=_\mathbb{R}$ in our system. As all the premises are $\forall$-formulas, we can apply Corollary 3.5 to extract a functional Φ such that for all $b \in \mathbb{N}$,

$$\forall \underline{u} \in P\, \forall \forall \underline{y} \in K\, \forall x, p \in X\, \forall T : X \to X\, \forall \Omega : \mathbb{N} \to \mathbb{N}$$
$$\Big(T\, x\text{-maj. mod. } \Omega \wedge d(x, p) \leq_\mathbb{R} b \wedge \forall k \leq \Phi(\underline{u}, b, \Omega, \eta)\big(d(p, Tp) \leq_\mathbb{R} 2^{-k}\big)$$
$$\wedge \forall n \leq \Phi(\underline{u}, b, \Omega, \eta)\, B_\forall \to \exists N \leq \Phi(\underline{u}, b, \Omega, \eta)\, C_\exists\Big),$$

that is

$$\forall \underline{u} \in P\, \forall \underline{y} \in K\, \forall x \in X\, \forall T : X \to X\, \forall \Omega : \mathbb{N} \to \mathbb{N}\Big(T\, x\text{-maj. mod. } \Omega \wedge$$
$$\exists p \in X\big(d(x, p) \leq b \wedge \forall k \leq \Phi(\underline{u}, b, \Omega, \eta)\big(d(p, Tp) \leq 2^{-k}\big)\big) \wedge$$
$$\forall n \leq \Phi(\underline{u}, b, \Omega, \eta)\, B_\forall \to \exists N \leq \Phi(\underline{u}, b, \Omega, \eta)\, C_\exists\Big).$$

Use the fact that the existence of $p \in X$ such that $d(x,p) \leq b$ and $\forall k \leq \Phi\big(d(p,Tp) \leq 2^{-k}\big)$ is equivalent with the existence of $p \in X$ such that $d(x,p) \leq b$ and $d(p,Tp) \leq 2^{-\Phi}$ which is obviously implied by $\forall \delta > 0\, (Fix_\delta(T,x,b) \neq \emptyset)$. $\square$

We shall apply the above corollary in the next section for nonexpansive mappings $T : X \to X$. In this case, as we have seen in Lemma 3.2, a modulus of majorizability at x is given by $\Omega(n) = n + \tilde{b}$, where $\tilde{b} \geq d(x,Tx)$, so the bound Φ will depend on $\underline{u}, \eta, b$ and $\tilde{b} > 0$ such that $d(x,Tx) \leq \tilde{b}$.

For all $\delta > 0$ there exists $y \in X$ such that $Fix_\delta(T,x,b) \neq \emptyset$, hence

$$d(x,Tx) \leq d(x,y) + d(y,Ty) + d(Ty,Tx) \leq 2d(x,y) + d(y,Ty) \leq 2b + \delta$$

for all $\delta > 0$. It follows that $d(x,Tx) \leq 2b$, so we can take $\tilde{b} := 2b$. As a consequence, the bound Φ will depend only on $\underline{u}, b$ and η.

The above logical metatheorems were obtained for classical proofs in metric, W-hyperbolic or UCW-hyperbolic spaces. Gerhardy and Kohlenbach [7] considered similar metatheorems for semi-intuitionistic proofs, that is proofs in intuitionistic analysis enriched with some non-constructive principles. Let $\mathcal{A}^\omega_i := \mathbf{E} - \mathbf{HA}^\omega + \mathbf{AC}$, where $\mathbf{E} - \mathbf{HA}^\omega$ is the extensional Heyting arithmetic in all finite types and $\mathbf{AC}$ is the full axiom of choice. The theories $\mathcal{A}^\omega_i[X,d]_{-b}$, $\mathcal{A}^\omega_i[X,d,W]_{-b}$ and $\mathcal{A}^\omega_i[X,d,UCW]_{-b}$ are obtained as above as extensions of $\mathcal{A}^\omega_i$. We refer to [7] for details.

Comprehension for negated formulas is the following principle:

$$CA^{\underline{\rho}}_{\neg} \quad \exists \Phi \leq_{\underline{\rho} \to \mathbb{N}} \lambda\, \underline{x}^{\underline{\rho}}.1^{\mathbb{N}}\, \forall \underline{y}^{\underline{\rho}} \big(\Phi(\underline{y}) =_{\mathbb{N}} 0 \leftrightarrow \neg A(\underline{y}\big),$$

where $\underline{\rho} = \rho_1, \ldots, \rho_k$ and $\underline{y} = y_1^{\rho_1}, \ldots, y_k^{\rho_k}$.

The following result is an adaptation to UCW-spaces of [7, Corollary 4.9].

Proposition 3.8. *Let P be $\mathbb{N}, \mathbb{N} \to \mathbb{N}$ or $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ and K be an $\mathcal{A}^\omega_i$-definable compact Polish space. Let $B(\underline{u}, \underline{y}, T, x)$ and $C(\underline{u}, \underline{y}, T, x, N)$ be arbitrary, but not containing 0_X.*

Assume that $\mathcal{A}^\omega_i[X,d,UCW,\eta]_{-b} + CA_\neg$ proves that

$$\forall \underline{u} \in P\, \forall \underline{y} \in K\, \forall x \in X\, \forall T : X \to X\, \forall \Omega : \mathbb{N} \to \mathbb{N}$$
$$\Big(T \textit{ is } x\textit{-majorizable with modulus } \Omega \wedge \neg B \to \exists N \in \mathbb{N}\, C\Big).$$

Then one can extract a primitive recursive in the sense of Gödel functional Φ such that for all $\underline{u} \in P$ and $\Omega : \mathbb{N} \to \mathbb{N}$,

$$\forall \underline{y} \in K\, \forall x \in X\, \forall T : X \to X \exists N \leq \Phi(\underline{u}, \Omega, \eta)$$
$$\Big(T \textit{ is } x\textit{-majorizable with modulus } \Omega \wedge \neg B \to C\Big).$$

holds in any nonempty UCW-hyperbolic space (X,d,W,η).

As before, instead of a single premise B, we may have a finite conjunction of premises.

4 Logical discussion of the asymptotic regularity proof

By an inspection of the proof of Theorem 1.1, one can see that it consists of three important steps. One proves first the following result.

Proposition 4.1. *Let (X, d, W, η) be a UCW-hyperbolic space, $C \subseteq X$ be a convex subset and $T : C \to C$ nonexpansive with $Fix(T) \neq \emptyset$. Assume that*

$$\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) \text{ diverges} \quad \text{and} \quad \limsup_{n\to\infty} s_n < 1.$$

Then $\liminf\limits_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$.

A first remarks is that it is enough to consider nonexpansive mappings $T : X \to X$, as convex subsets of UCW-hyperbolic spaces are themselves UCW-hyperbolic spaces.

The assumption that $\sum\limits_{n=0}^{\infty} \lambda_n(1-\lambda_n)$ diverges is equivalent with the existence of a rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$ for the series. As (s_n) is a sequence in $[0, 1]$, the assumption that $\limsup\limits_{n\to\infty} s_n < 1$ is equivalent with the existence of $L, N_0 \in \mathbb{N}$ such that $s_n \leq 1 - \dfrac{1}{L}$ for all $n \geq N_0$.

Furthermore, since $d(x_n, Tx_n) \geq 0$, the following statements are equivalent:

(i) $\liminf\limits_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) for all $k, l \in \mathbb{N}$ there exists $N \geq k$ such that $d(x_N, Tx_N) < 2^{-l}$.

By a *modulus of liminf* Δ for $(d(x_n, Tx_n))$ we shall understand a mapping $\Delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ giving for each $k, l \in \mathbb{N}$ an N as in (ii) above.

One can easily conclude that $\mathcal{A}^\omega[X, d, UCW, \eta]_{-b}$ proves the following formalized version of Proposition 4.1:

$$\forall k, l, N_0, L \in \mathbb{N},\ \forall \theta : \mathbb{N} \to \mathbb{N}\ \forall \lambda_{(\cdot)}^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})}, s_{(\cdot)}^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})}\ \forall x \in X, T : X \to X$$
$$\Big(Fix(T) \neq \emptyset \wedge B_\forall \to \exists N \in \mathbb{N}\big(N \geq_0 k + N_0 \wedge d_X(x_N, Tx_N) <_{\mathbb{R}} 2^{-l}\big)\Big),$$

where $\lambda_{(\cdot)}^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})}, s_{(\cdot)}^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})}$ represent elements of the compact Polish space $[0, 1]^\infty$ with the product metric and

$$\begin{aligned} B_\forall \quad \equiv \quad & T \textit{ nonexpansive} \wedge \forall n \in \mathbb{N} \left(\sum_{i=0}^{\theta(n)} \lambda_i(1-\lambda_i) \geq_{\mathbb{R}} n\right) \wedge \\ & \forall n \in \mathbb{N} \ \left(n \geq_0 N_0 \to s_n \leq_{\mathbb{R}} 1 - \frac{1}{L}\right). \end{aligned}$$

Using the representation of real numbers in our system, one can see immediately that $B_\forall$ is a universal formula. Corollary 3.7 and the discussion afterwards yield the extractability of a functional $\Delta := \Delta(l,k,b,N_0,L,\theta,\eta)$ such that for all $k,b,N_0,L,\theta,(\lambda_n),(s_n)$,

$$\forall x \in X, T : X \to X \big(\forall \delta > 0 (Fix_\delta(T,x,b) \neq \emptyset) \wedge B_\forall \to \\ \forall l \in \mathbb{N} \exists N \leq \Delta \ (N \geq_0 k \wedge d(x_N, T(x_N)) <_\mathbb{R} 2^{-l})\big)$$

holds in any nonempty UCW-hyperbolic space (X,d,W,η). As a consequence, Δ is a modulus of liminf for $(d(x_n,T(x_n)))$.

For the rest of the proof one uses the assumption that the series $\sum\limits_{n=0}^{\infty} s_n(1-\lambda_n)$ converges, which is equivalent with the fact that the sequence of partial sums $\alpha_n := \sum\limits_{i=0}^{n} s_i(1-\lambda_i)$ is bounded.

In the second step of the proof, upper bounds for (α_n) and $d(x,Tx)$ are used to compute an upper bound for the sequence $(d(x_n,Tx_n))$.

The third and final step of the proof is the following result.

Proposition 4.2. *Assume that* $\liminf\limits_{n\to\infty} d(x_n,Tx_n) = 0$ *and that* $\sum\limits_{n=0}^{\infty} s_n(1-\lambda_n)$ *diverges. Then* $\lim\limits_{n\to\infty} d(x_n,Tx_n) = 0$.

The proof of these last two steps is fully constructive and one can easily see that the formal system $\mathcal{A}_i^\omega[X,d,UCW,\eta]_{-b}$ proves

$$\forall l, M \in \mathbb{N} \forall \gamma : \mathbb{N} \to \mathbb{N} \forall \Delta : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \forall \lambda_{(\cdot)}, s_{(\cdot)} \ \forall x \in X, T : X \to X \\ \big(T \text{ nonexpansive } \wedge A \to \exists N \in \mathbb{N}\, C\big),$$

where $C \equiv \forall n \in \mathbb{N} \left(d_X(x_{n+N}, Tx_{n+N}) \leq_\mathbb{R} 2^{-l}\right)$ and

$$\begin{aligned} A \quad \equiv \quad & \Delta \text{ modulus of liminf for} (d(x_n,Tx_n)) \wedge M \text{ upper bound on } (\alpha_n) \\ & \wedge \gamma \text{ Cauchy modulus for } (\alpha_n). \end{aligned}$$

Since A is a universal, hence negated formula, we can apply Proposition 3.8 for T nonexpansive to conclude that we can extract a Gödel primitive recursive functional $\Phi := \Phi(l,M,\Delta,\gamma,b,\eta)$ such that for all $x \in X$, $T : X \to X$,

$$\exists N \leq \Phi \, (T \text{ nonexpansive } \wedge d(x,Tx) \leq_\mathbb{R} b \wedge A \to C)$$

holds in any nonempty UCW-hyperbolic space (X,d,W,η). Hence, there exists $N \leq \Phi$ such that $d(x_{n+N}, Tx_{n+N}) \leq 2^{-l}$ for all $n \in \mathbb{N}$. It follows that $d(x_n,Tx_n) \leq 2^{-l}$ for all $n \geq \Phi$.

Thus, Φ is a rate of asymptotic regularity for the Ishikawa iteration (x_n).

5 Proof of Theorem 2.1

In the following, (X, d, W, η) is a UCW-hyperbolic space, $C \subseteq X$ is a nonempty convex subset, $T : C \to C$ is a nonexpansive mapping and $(\lambda_n), (s_n)$ are sequences in $[0, 1]$.

Given $x \in C$, (x_n) is the Ishikawa iteration starting with x, defined by (3). We shall use the following notation

$$y_n := (1 - s_n)x_n \oplus s_n T x_n.$$

Thus, $x_{n+1} = (1 - \lambda_n)x_n \oplus \lambda_n T y_n$.

We recall first a very useful property of UCW-hyperbolic spaces.

Lemma 5.1. *[26] Let (X, d, W, η) be a UCW-hyperbolic space. Assume that $r > 0, \varepsilon \in (0, 2]$ and $a, x, y \in X$ are such that*

$$d(x, a) \le r,\ d(y, a) \le r \text{ and } d(x, y) \ge \varepsilon r.$$

Then for any $\lambda \in [0, 1]$ and for all $s \ge r$,

$$d((1 - \lambda)x \oplus \lambda y, a) \le (1 - 2\lambda(1 - \lambda)\eta\,(s, \varepsilon))\, r.$$

The following lemma collects properties of the Ishikawa iteration, which will be needed in the sequel.

Lemma 5.2. *For all $n \in \mathbb{N}$ and $x, z \in X$, the following hold*

$$\begin{aligned}
(1 - s_n)d(x_n, Tx_n) &\le d(x_n, Ty_n) && (8)\\
d(x_{n+1}, Tx_{n+1}) &\le (1 + 2s_n(1 - \lambda_n))d(x_n, Tx_n) && (9)\\
d(y_n, z) &\le d(x_n, z) + d(z, Tz) && (10)\\
d(Ty_n, z) &\le d(x_n, z) + 2d(z, Tz) && (11)\\
d(x_{n+1}, z) &\le d(x_n, z) + 2\lambda_n d(z, Tz) && (12)\\
d(x_n, z) &\le d(x, z) + 2\sum_{i=0}^{n-1} \lambda_i d(z, Tz) && (13)\\
&\le d(x, z) + 2nd(z, Tz) && (14)
\end{aligned}$$

Proof. (8) and (9) are proved in [26, Lemma 4.1].

$$\begin{aligned}
d(y_n, z) &\le (1 - s_n)d(x_n, z) + s_n d(Tx_n, z)\\
&\le (1 - s_n)d(x_n, z) + s_n d(Tx_n, Tz) + s_n d(Tz, z)\\
&\le d(x_n, z) + d(Tz, z).\\
d(Ty_n, z) &\le d(Ty_n, Tz) + d(Tz, z) \le d(y_n, z) + d(z, Tz)\\
&\le d(x_n, z) + 2d(z, Tz)\\
d(x_{n+1}, z) &\le (1 - \lambda_n)d(x_n, z) + \lambda_n d(Ty_n, z)\\
&\le (1 - \lambda_n)d(x_n, z) + \lambda_n\big(d(x_n, z) + 2d(z, Tz)\big)\\
&\le d(x_n, z) + 2\lambda_n d(z, Tz).
\end{aligned}$$

(14) follows easily by induction. □

Proposition 5.3. *Assume that the following hold.*

(i) $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n)$ *with rate of divergence* $\theta : \mathbb{N} \to \mathbb{N}$*;*

(ii) $\limsup_n s_n < 1$ *with* $L, N_0 \in \mathbb{N}$ *satisfying* $s_n \leq 1 - \dfrac{1}{L}$ *for all* $n \geq N_0$*.*

Let $x \in C, b > 0$ *be such that for any* $\delta > 0$ *there is* $z \in C$ *with*

$$d(x,z) \leq b, \quad \textit{and} \;\; d(z,Tz) < \delta. \tag{15}$$

Then

$$\forall \varepsilon > 0, k \in \mathbb{N}\, \exists N \in [k+N_0, \Delta] \left(d(x_N, Tx_N) < \varepsilon\right),$$

where

$$\Delta := \Delta(\varepsilon, k, \eta, b, N_0, L, \theta) \quad = \quad \theta(P + k + N_0),$$

with

$$P \quad = \quad \left\lceil \frac{L(b+1)}{\varepsilon \cdot \eta\left(b+1, \dfrac{\varepsilon}{L(b+1)}\right)} \right\rceil.$$

Proof. Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Using (8) and (ii) one can easily see that it suffices to prove that

$$\exists N \in [k+N_0, \Delta] \left(d(x_N, Ty_N) < \frac{\varepsilon}{L}\right). \tag{16}$$

Assume by contradiction that (16) does not hold, hence $d(x_n, Ty_n) \geq \dfrac{\varepsilon}{L}$ for all $n \in [k+N_0, \Delta]$.

We shall use the notations

$$\delta := \frac{1}{4(\Delta+1)} \quad \text{and} \quad a_n := d(x_n, z) + 2d(z, Tz).$$

Let $z \in C$ satisfy (15) for δ defined above. As a consequence of (14), we get that for all $n \in [k+N_0, \Delta]$

$$\begin{aligned} a_n \quad &\leq \quad d(x,z) + (2n+2)d(z,Tz) \leq b + (2n+2)d(z,Tz) \leq b + 2(\Delta+1)\delta \\ &< \quad b+1. \end{aligned}$$

Since

$$\begin{aligned} d(Ty_n, z) \quad &\leq \quad a_n \quad \text{by (11)}, \\ d(x_n, Ty_n) \quad &\geq \quad \frac{\varepsilon}{L} \geq \frac{\varepsilon}{L(b+1)} \cdot a_n, \text{ and} \\ 0 < \frac{\varepsilon}{L(b+1)} \quad &\leq \quad \frac{d(x_n, Ty_n)}{b+1} \leq \frac{d(x_n,z) + d(Ty_n,z)}{b+1} \leq \frac{2a_n}{b+1} \\ &\leq \quad 2, \end{aligned}$$

we can apply Lemma 5.1 with $r := a_n$, $s := b+1$ and $\dfrac{\varepsilon}{L(b+1)}$ to obtain

$$\begin{aligned}
d(x_{n+1}, z) &= d((1-\lambda_n)x_n \oplus \lambda_n T y_n, z) \\
&\leq \left(1 - 2\lambda_n(1-\lambda_n)\eta\left(b+1, \frac{\varepsilon}{L(b+1)}\right)\right) a_n \\
&= d(x_n, z) + 2d(z, Tz) - 2\lambda_n(1-\lambda_n)\eta\left(b+1, \frac{\varepsilon}{L(b+1)}\right) a_n
\end{aligned}$$

As $a_n \geq \dfrac{d(x_n, Ty_n)}{2} \geq \dfrac{\varepsilon}{2L}$, we get that for all $n \in [k+N_0, \Delta]$,

$$d(x_{n+1}, z) \leq d(x_n, z) + 2d(z, Tz) - \frac{\varepsilon}{L}\lambda_n(1-\lambda_n)\eta\left(b+1, \frac{\varepsilon}{L(b+1)}\right). \tag{17}$$

Adding (17) for $n = k+N_0, \ldots, \Delta$, it follows that

$$\begin{aligned}
d(x_{\Delta+1}, z) \leq\ & d(x_{k+N_0}, z) + 2(\Delta - k - N_0 + 1)d(z, Tz) - \\
& -\frac{\varepsilon}{L}\eta\left(b+1, \frac{\varepsilon}{L(b+1)}\right) \sum_{n=k+N_0}^{\Delta} \lambda_n(1-\lambda_n).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{n=k+N_0}^{\Delta} \lambda_n(1-\lambda_n) &= \sum_{n=0}^{\theta(P+k+N_0)} \lambda_n(1-\lambda_n) - \sum_{n=0}^{k+N_0-1} \lambda_n(1-\lambda_n) \\
&\geq (P+k+N_0) - (k+N_0) = P,
\end{aligned}$$

it follows that

$$\begin{aligned}
d(x_{\Delta+1}, z) &\leq d(x_{k+N_0}, z) + 2(\Delta - k - N_0 + 1)d(z, Tz) - \\
&\quad -\frac{P\varepsilon}{L}\eta\left(b+1, \frac{\varepsilon}{L(b+1)}\right) \\
&\leq d(x, z) + 2(\Delta+1)d(z, Tz) - \frac{P\varepsilon}{L}\eta\left(b+1, \frac{\varepsilon}{L(b+1)}\right) \text{ by (14)} \\
&\leq b + \frac{1}{2} - (b+1) < 0,
\end{aligned}$$

that is a contradiction. $\square$

The following lemma on sequences of real numbers is also used in the proof.

Lemma 5.4. *Let $(a_n), (t_n)$ be sequences of nonnegative reals satisfying*

$$a_{n+1} \leq (1+t_n)a_n \quad \textit{for all } n \in \mathbb{N}. \tag{18}$$

(i) For all $m, n \in \mathbb{N}$, $a_{m+n} \leq \displaystyle\prod_{j=m}^{m+n-1}(1+t_j)a_m$.

(ii) Assume that $\sum_{n=0}^{\infty} t_n \leq M$. *Then* $a_n \leq a_0 \exp(M)$ *for all* $n \in \mathbb{N}$.

Proof. (i) By induction on n.

(ii) Applying (i) with $m = 0$ one gets

$$\begin{aligned} a_n &\leq \prod_{j=0}^{n-1}(1+t_j)a_0 = \exp\left(\ln\left(\prod_{j=0}^{n-1}(1+t_j)\right)\right)a_0 \\ &\leq \exp\left(\sum_{j=0}^{n-1} t_j\right)a_0, \quad \text{since } \ln(1+x) \leq x \text{ for } x \geq 0 \\ &\leq a_0 \exp(M). \end{aligned}$$

□

5.1 Proof of Theorem 2.1

We are ready now to prove the main result of the paper. Let $\varepsilon > 0$, $b > 0$, $x \in C$, $(\lambda_n), (s_n), \theta, \gamma, L, N_0, M, \gamma_0$ be as in the hypothesis.

Denote for simplicity $\alpha_n := \sum_{i=0}^{n} s_i(1-\lambda_i)$. Applying (9), the hypothesis (iii) and Lemma 5.4, we get that for all $n \in \mathbb{N}$,

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x, Tx)\exp(2M) \leq b\exp(2M) \quad \text{and} \\ d(x_{n+1}, Tx_{n+1}) &\leq d(x_n, Tx_n) + 2b\exp(2M)s_n(1-\lambda_n). \end{aligned}$$

It follows that for all $m \in \mathbb{N}, n \geq 1$,

$$\begin{aligned} d(x_{n+m}, Tx_{n+m}) &\leq d(x_n, Tx_n) + 2b\exp(2M)\sum_{i=n}^{n+m-1} s_i(1-\lambda_i) \\ &= d(x_n, Tx_n) + 2b\exp(2M)(\alpha_{n+m-1} - \alpha_{n-1}). \end{aligned}$$

An application of Proposition 5.3 with $\frac{\varepsilon}{2}$ and $k := \gamma_0 + 1$ gives us an $N \in \mathbb{N}$ such that $d(x_N, Tx_N) < \frac{\varepsilon}{2}$ and

$$\begin{aligned} \gamma_0 + 1 + N_0 \leq N &\leq \Delta\left(\frac{\varepsilon}{2}, \gamma_0 + 1, \eta, b, N_0, L, \theta\right) \\ &= \Phi(\varepsilon, \eta, b, N_0, L, \theta, \gamma, M). \end{aligned}$$

Let $n \geq \Phi$ be arbitrary. Since $n \geq N$, we have that $n = N + l$ for some

$l \in \mathbb{N}$. It follows that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_N, Tx_N) + 2b\exp(2M)(\alpha_{N+l-1} - \alpha_{N-1}) \\ &= d(x_N, Tx_N) + 2b\exp(2M)\,(\alpha_{\gamma_0+N_0+q+l} - \alpha_{\gamma_0+N_0+q}) \\ &\quad \text{where } q = N - 1 - \gamma_0 - N_0 \geq 0 \\ &\leq \frac{\varepsilon}{2} + 2b\exp(2M)\,(\alpha_{\gamma_0+N_0+q+l} - \alpha_{\gamma_0}) \\ &< \frac{\varepsilon}{2} + 2b\exp(2M) \cdot \frac{\varepsilon}{4b\exp(2M)} = \varepsilon, \end{aligned}$$

since γ is a Cauchy modulus for (a_n).

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